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# A local energy-transfer theory of isotropic turbulence

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**Abstract.** In this paper we show that a nonlinear integral equation for turbulent energy transport may be reinterpreted in terms of a Heisenberg-type effective viscosity. A new equation is derived for the effective viscosity. This is found to permit general expansions of the integral kernels, in powers of wavenumber ratios, leading to local (differential) equations for the energy spectrum and effective viscosity. It is found that these equations yield the Kolmogoroff distribution as the inertial-range solution, and that the numerical predictions agree quite well with experimental results. The final equations are similar to equations recently derived by Nakano, and the relationship between the two theories is discussed.

## 1. Introduction

A local differential equation for the energy spectrum in isotropic turbulence has recently been developed (Edwards and McComb 1971) and subsequently applied, in generalized form, to a simple turbulent shear flow (Edwards and McComb 1972). The basis of this work was a rather phenomenological treatment of the turbulent response function, in which the mode lifetime (or effective viscosity) was taken to have known forms in the inertial and dissipation ranges of wavenumbers. The main aim of the theory presented in these papers was to simplify existing formalism, with a view to practical applications. Thus, although the final equations were found to be relatively simple and qualitatively quite good, no attempt was made at numerical predictions.

In the present paper, a new equation is obtained for the turbulent effective viscosity. It is found to provide the basis for a more general expansion method (than those referred to above). This leads to a local differential equation for the energy spectrum, which appears to be sufficiently accurate for numerical calculations. The work described is also found to be quite closely related to a recent theory by Nakano (1972) and the two theories are compared in § 6. We begin by briefly reviewing the basic problem and some relevant theoretical developments.

Let us consider isotropic turbulence in an incompressible fluid, which occupies a box of side  $L$ . If we let the velocity field be  $u_\alpha(x, t)$  then the Fourier components of this are defined by

$$u_\alpha(x, t) = \sum_{\mathbf{k}} u_\alpha(\mathbf{k}, t) e^{i\mathbf{k}\cdot\mathbf{x}}. \quad (1.1)$$

These satisfy the Fourier-transformed Navier–Stokes equation thus:

$$\left( \frac{\partial}{\partial t} + \nu k^2 \right) u_\alpha(\mathbf{k}, t) = \sum_{\mathbf{j}} M_{\alpha\beta\gamma}(\mathbf{k}) u_\beta(\mathbf{j}, t) u_\gamma(\mathbf{k} - \mathbf{j}, t) \quad (1.2)$$

along with the continuity equation,

$$k_\alpha u_\alpha(\mathbf{k}, t) = 0 \quad (1.3)$$

(eg see Batchelor 1959). We note that an arbitrary forcing term may be added to (1.2) in order to sustain the turbulence. Also, the inertial transfer operator  $M_{\alpha\beta\gamma}(\mathbf{k})$  is given by

$$M_{\alpha\beta\gamma}(\mathbf{k}) = \frac{1}{2}i(k_\beta D_{\alpha\gamma}(\mathbf{k}) + k_\gamma D_{\alpha\beta}(\mathbf{k})) \quad (1.4)$$

where

$$D_{\alpha\beta}(\mathbf{k}) = \delta_{\alpha\beta} - k_\alpha k_\beta |\mathbf{k}|^{-2}. \quad (1.5)$$

In order to pursue a statistical treatment, we introduce the correlation of two velocities:

$$\left(\frac{L}{2\pi}\right)^3 \langle u_\alpha(\mathbf{k}, t + \tau) u_\beta(\mathbf{k}, \tau) \rangle = D_{\alpha\beta}(\mathbf{k}) q_k(t) \quad (1.6)$$

where the form of (1.6) is dictated by isotropy. We note that  $k_\alpha D_{\alpha\beta}(\mathbf{k}) = 0$ , so that (1.6) satisfies the continuity equation; and that  $q_k(t)$  is a scalar function, which depends only on the magnitude of  $\mathbf{k}$ .

The basic problem of turbulence is now plain: if we form an equation for  $q_k(t)$  from (1.2), the result involves the unknown triple correlation,  $\langle u_\beta u_\gamma u_\alpha \rangle$ . Similarly, the equation for the third-order correlation must contain the unknown fourth-order correlation. This process may be continued to any order and the problem is to obtain a tractable closed set of equations for  $q_k$ .

There have been many phenomenological and semi-empirical approaches to this subject (eg see Hinze 1959). In general, the most fruitful concept has been to assume that the effect of the nonlinear term in (1.2) may be represented by an effective eddy viscosity acting on the velocity field. The simplest and most successful (from a fundamental point of view) of these theories is probably that due to Heisenberg (1948), who assumed that an eddy of given size could only lose energy to eddies of smaller size. That is, the effective viscosity for mode  $\mathbf{k}$ ,  $\nu_k$  (say), would only involve a sum over wavenumbers greater than  $\mathbf{k}$ . From dimensional considerations he deduced an energy equation

$$\int_0^k \frac{\partial E(k)}{\partial t} = -2(\nu + \nu_k) \int_0^k E(k) k^2 dk \quad (1.7)$$

where

$$\nu_k = A \int_k^\infty \left(\frac{E(j)}{j^3}\right)^{1/2} dj, \quad (1.8)$$

$A$  being a constant, and

$$E(k) = 4\pi k^2 q_k, \quad (1.9)$$

is the energy spectrum. In the limit  $\nu \rightarrow 0$ , the steady-state solution of (1.9) may be shown to be  $E(k) \sim k^{-5/3}$ , which is the well known Kolmogoroff distribution (see Batchelor 1959).

Although equation (1.7) is nonlocal in form, the solution it yields was originally deduced by Kolmogoroff on the basis of similarity principles, which were equivalent to an assumption that energy transfers are local in wavenumber.

The first similarity principle stated that the statistical distribution of energy in the eddies can only depend on the rate of energy input (per unit mass) and the kinematic

viscosity. The second principle stated that there will be an inertial range of wavenumbers, where the spectrum will be independent of  $\nu$ , provided the Reynolds number is large enough.

Dimensional analysis then gave the form

$$E(k) = \alpha \epsilon^{2/3} k^{-5/3} \quad (1.10)$$

where  $\alpha$  is a constant and  $\epsilon$  is the rate of energy input (or dissipation) per unit mass of fluid. This result has received considerable experimental confirmation and is now widely accepted. The physical picture it suggests for the turbulent energy transfer process is appealingly simple.

However, in recent years, theories of turbulence have moved away from such simplicity and have invoked the methods of modern statistical mechanics and field theory. The pioneering work in this area is due to Kraichnan (1959), who introduced a turbulent response function, relating infinitesimal changes in the forces driving the turbulence to resulting changes in the velocity field. An iterative solution of the equations of motion led to coupled integro-differential equations for the energy spectrum and mean response function. This work is known as the direct-interaction approximation.

On the other hand, Edwards (1964) studied the linear problem of obtaining the probability distribution of fluctuating velocities by deriving a generalized Liouville equation. This was approximated by an expression with lowest-order operator of the Fokker–Planck form and the distribution functional obtained. In this case the additional quantity needed for closure was found to arise quite naturally as a dynamical friction term (or effective viscosity) in the Fokker–Planck equation.

It was pointed out by Kraichnan (1964a) that, if the direct-interaction response function and energy spectrum were approximated by an exponential decay, with the effective viscosity as modal lifetime, then time-independent equations very similar to those of Edwards could be obtained. In fact the two energy equations are identical but there is a difference in the equations for the effective viscosity. This result was also obtained by Herring (1965), who used a self-consistent field method to calculate the energy spectrum.

A reasonable index of performance for these theories may be obtained by using them to calculate the Kolmogoroff constant  $\alpha$ , in (1.10), and compare the result with an experimental value. (Currently this aspect of turbulence is the subject of some debate: see eg Kraichnan 1973; but we shall merely take the pragmatic view that (1.10), or something very close to this, has been found experimentally). The result for both the direct-interaction and generalized Fokker–Planck theories is that an infinite value is found for  $\alpha$ . This is due to a divergence at the origin in the wavenumber integral for the effective viscosity.

A physical interpretation of this has been given by Kraichnan (1964b), who argues that, if transfers are to be local, the large eddies will convect small eddies without significant distortion or energy transfer. As the mathematical formalism takes no account of this separation of length scales, the effect of large eddies (small  $k$ ) is overestimated.

Kraichnan (1964b, 1965) has made two attempts to rectify this situation. In the first, a modified form of Navier–Stokes equation (from which all interactions, involving a given spatial scale with much larger scales, are removed) is studied. A value of  $\alpha$ , which depends on an arbitrary cut-off ratio, is obtained.

In the second method, a quasi-lagrangian coordinate system is used and application of the direct-interaction approximation leads to a set of equations which yield the Kolmogoroff distribution as the inertial range solution. Unfortunately these equations

are extremely complicated and, in order to permit numerical integration, have to be abridged in what seems a rather arbitrary way.

This problem has also been studied by Edwards and McComb (1969) who obtained an equation for the effective viscosity by choosing it such that the turbulent entropy was maximized. While this method also yields the  $k^{-5/3}$  law as a solution, it is also rather complicated and moreover, gives an unacceptably large value for  $\alpha$ .

The formal mathematical structure of the above theories has received a certain amount of additional confirmation, eg Wyld (1961), Balescu and Senatorski (1970). However, the problem of defining an effective viscosity such that a reasonably simple set of equations may be obtained (and an accurate calculation of the Kolmogoroff distribution made) remains an obstinate bar to further progress.

Recently this problem has been tackled by Nakano (1972), who derives a differential equation for the energy spectrum, from a direct expansion of the Navier–Stokes equation. Although this work is based on arguments similar to those of Kraichnan (1964b) and, likewise, contains arbitrary bounds on the wavenumber integrals, it does possess a number of interesting features. We shall discuss this theory in some detail, later in the present paper.

While the work described in this paper also involves wavenumber cut-off ratios, it will be seen that these arise naturally from the mathematics and are not just based on intuition about the physics of the turbulence.

## 2. The effective viscosity

We take as our starting point the equation for  $q_k$ , which was derived by Edwards (1964):

$$\frac{\partial q_k}{\partial t} = h_k - 2\nu k^2 q_k + 2 \int d^3j \int d^3l \frac{\delta_{kjl} L_{kjl} q_j (q_l - q_k)}{\omega_k + \omega_j + \omega_l}, \quad (2.1)$$

where  $h_k$  is an arbitrary energy input, to drive the turbulence:  $\delta_{kjl} = 1$  if  $\mathbf{k} + \mathbf{j} + \mathbf{l} = 0$  but zero otherwise and  $\omega_k$  is the lifetime of mode  $k$  and is related to the effective viscosity by

$$\omega_k = (\nu + \nu_k) k^2. \quad (2.2)$$

Strictly, equation (2.1) is in steady-state form and  $\partial q_k / \partial t$  can only describe variations in external conditions but not, for example, the turbulent decay. We retain this term for the moment as it simplifies the interpretation of (2.1): later we shall set it equal to zero.

Equation (2.1) may then be put in words as:

total change of energy in mode  $k$  = rate of doing work on mode  $k$   
– viscous dissipation in mode  $k$  + net transfer into  $k$ , from other modes.

The latter may be either positive or negative. Calling this term  $T_k$ , we may express conservation of energy in the form:

$$\int d^3k T(k) = 2 \int d^3k \int d^3j \int d^3l \frac{\delta_{kjl} L_{kjl} q_j (q_l - q_k)}{\omega_k + \omega_j + \omega_l} = 0, \quad (2.3)$$

this result being due to the cancellation of the two parts of the integrand. We will find it

convenient to integrate over  $l$ , thus removing the delta function, and obtain

$$T(k) = 2 \int d^3j \frac{L_{kj} q_j (q_{|k+j|} - q_k)}{\omega_k + \omega_j + \omega_{|k+j|}}, \quad (2.4)$$

where

$$L_{kj} = \frac{(k^4 + 2k^3j\mu - kj^3\mu)(1 - \mu^2)}{k^2 + 2kj\mu + j^2} \quad (2.5)$$

$\mu$  being the cosine of the angle between the vectors  $\mathbf{k}$  and  $\mathbf{j}$ .  $T(k)$ , as given by (2.4), still satisfies equation (2.3).

Edwards (1964) interpreted  $T(k)$  as being the difference between input and output terms and introduced the diffusion coefficient  $S(k)$  and dynamical friction  $R(k)$ , such that

$$S(k) = 2 \int d^3j \frac{L_{kj} q_j q_{|k+j|}}{\omega_k + \omega_j + \omega_{|k+j|}} \quad (2.6)$$

$$R(k) = \int d^3j \frac{L_{kj} q_j}{\omega_k + \omega_j + \omega_{|k+j|}}. \quad (2.7)$$

Hence  $T(k)$  could be written as

$$T(k) = S(k) - 2R(k)q_k, \quad (2.8)$$

so that equation (2.1) became

$$\frac{\partial q_k}{\partial t} = h_k + S(k) - 2\omega_k q_k, \quad (2.9)$$

where

$$\omega_k = \nu k^2 + R(k). \quad (2.10)$$

Thus  $R(k)$  could be interpreted as  $k^2$  times the turbulent viscosity.

Similarly, the direct-interaction approximation, when put in an exponential decay framework, leads to equations (2.1), (2.4) and (2.5) for  $q_k$  but differs from equation (2.7) for  $R(k)$ , in that  $\omega_k$  does not appear in the denominator. As we pointed out in the introduction, these theories do not lead to the Kolmogoroff distribution, as the integral in (2.7) is divergent at  $j \sim 0$ .

Although (2.6), (2.7) and (2.10) seem a natural interpretation of  $T(k)$ , Herring (1965) has commented that there are many ways of choosing the two functions  $S(k)$  and  $R(k)$ , such that the appropriate constraints are satisfied. It has also been pointed out (Edwards and McComb 1969), that  $T(k)$  is only convergent at  $j \sim 0$  because of the cancellation of separately divergent terms. Thus, near  $j \sim 0$ , the modes are not really independent and the separation into  $S(k)$  and  $R(k)q_k$  becomes meaningless: only the net change is well defined. It is possible both to meet this latter point, and to achieve a close correspondence with the physical ideas of Heisenberg, by defining the effective viscosity in a new way. We now do this as follows.

Let us introduce a wavenumber  $K$ , such that

$$\int_{k \leq K} d^3k h_k = \epsilon = -2\nu \int_{k \geq K} d^3k k^2 q_k, \quad (2.11)$$

where  $\epsilon$  is the rate of energy dissipation per unit mass of fluid. In other words,  $K$  lies in the inertial range of wavenumbers, but is otherwise arbitrary. Consider equation (2.1),

with  $\partial q_k / \partial t = 0$  (steady state): if we integrate the right-hand side with respect to  $k$ , we have

$$\int d^3k (h_k - 2\nu k^2 q_k + T(k)) = 0. \quad (2.12)$$

Let us split up this integration at  $k = K$  to obtain

$$\int_{k \leq K} d^3k h_k - \int_{k \geq K} 2\nu k^2 q_k d^3k + \int_{k \leq K} d^3k T(k) + \int_{k \geq K} d^3k T(k) = 0 \quad (2.13)$$

and hence from (2.11) (or indeed, directly from (2.3))

$$\int_{k \leq K} d^3k T(k) + \int_{k \geq K} d^3k T(k) = 0. \quad (2.14)$$

Further, from the antisymmetric structure of  $T(k)$  we may write this out in full as

$$\begin{aligned} \int_{k \leq K} d^3k T(k) + \int_{k \geq K} d^3k T(k) &= \left( \int_{k \leq K} d^3k \int_{j \geq K} d^3j + \int_{k \geq K} d^3k \int_{j \leq K} d^3j \right) \\ &\times \frac{2L_{kj} q_j (q_{|k+j|} - q_k)}{\omega_k + \omega_j + \omega_{|k+j|}} = 0. \end{aligned} \quad (2.15)$$

Then, from (2.11), (2.13) and (2.15) we have the steady-state energy balance in the form of two equations:

$$\int_{k \leq K} d^3k \left( h_k + 2 \int_{j \geq K} d^3j \frac{L_{kj} q_j (q_{|k+j|} - q_k)}{\omega_k + \omega_j + \omega_{|k+j|}} \right) = 0 \quad (2.16)$$

and

$$\int_{k \geq K} d^3k \left( 2 \int_{j \leq K} d^3j \frac{L_{kj} q_j (q_{|k+j|} - q_k)}{\omega_k + \omega_j + \omega_{|k+j|}} - 2\nu k^2 q_k \right) = 0. \quad (2.17)$$

The physical interpretation of this is that  $T(k)$  behaves as a *net* sink or source of energy, according to the value of  $k$ . At small  $k$  (large eddies)  $T(k)$  absorbs all the input energy. This energy is transferred (by a conservative process) to ever-increasing wavenumbers. At large  $k$  (small eddies), the energy is removed by the action of the kinematic viscosity.

If there were no inertial transfer, then we would simply have

$$\int d^3k (h_k - 2\nu k^2 q_k) = 0. \quad (2.18)$$

As it is,  $T(k)$  exists and equations (2.16) and (2.17) may be regarded as the low and high wavenumber forms of (2.18), respectively.

Now the integral of  $T(k)$  is zero when evaluated between any two wavenumbers in the inertial range, so we may write (2.14) in the generalized form

$$\int_{k \leq k_1} d^3k T(k) + \int_{k \geq k_2} d^3k T(k) = 0, \quad (2.19)$$

provided  $k_1, k_2$  lie in the inertial range.

Making use of this, and adopting the form of energy balance suggested by equation (2.18), we write (2.16) and (2.17) in the following way:

$$\int_{k \leq k_1} d^3 k (h_k - 2\nu_k k^2 q_k) = 0 \quad (2.20)$$

$$\int_{k \geq k_2} d^3 k (H(k) - 2\nu k^2 q_k) = 0, \quad (2.21)$$

where  $\nu_k$  (the effective viscosity) is given by

$$\nu_k = -k^{-2} \int_{j \geq k_1} d^3 j \frac{L_{kj} q_j (q_{|k+j|} - q_k)}{q_k (\omega_k + \omega_j + \omega_{|k+j|})} \quad (2.22)$$

and  $H(k)$  (the diffusive input term) is given by

$$H(k) = 2 \int_{j \leq k_2} d^3 j \frac{L_{kj} q_j (q_{|k+j|} - q_k)}{\omega_k + \omega_j + \omega_{|k+j|}}. \quad (2.23)$$

It will be seen later that  $k^2 \nu_k$  reduces to  $R(k)$ , as defined by Edwards, except at small wavenumbers, where the difference in form is of crucial importance.

We may note that as  $k_1$  and  $k_2$  are in the inertial range, but otherwise arbitrary, we could make the replacement  $k_1 = k_2 = k$  and all the arguments of this section would still apply. However, we should also note that for  $j \sim k$ , the integrals in both (2.22) and (2.23) are zero, by cancellation. Thus, for a finite contribution, we have  $k > j$  and  $k < j$ , respectively. We will find it convenient to put this on a more formal basis by setting

$$\begin{aligned} k_1 &= mk \\ k_2 &= nk, \end{aligned} \quad (2.24)$$

where  $m \geq 1$ ,  $n \leq 1$ .

Also, it can be shown that, in the inertial range, the output part of  $T(k)$  may be transformed into the input part (and vice versa) by letting  $k \rightarrow j^{-1}$ ,  $j \rightarrow k^{-1}$ . Hence,  $m$  and  $n$  must satisfy

$$m = n^{-1}. \quad (2.25)$$

Alternatively, this relationship may be justified by the physical arguments due to Nakano (see § 6).

Thus, to summarize, the energy equation is written in analogous form to (2.9):

$$\frac{\partial q_k}{\partial t} = h_k - 2\omega_k q_k + H(k), \quad (2.26)$$

but now  $\omega_k$  is given by

$$\omega_k = (\nu + \nu_k) k^2, \quad (2.27)$$

where  $\nu_k$  is to be obtained from (2.22) and  $H(k)$ , which is given by (2.23), replaces  $S(k)$ .

Finally, in this section, we consider the question of how far the present theory may be applied outside the inertial range.

For the energy-containing range of wavenumbers, the answer to this is by no means obvious. However, in this range, the spectrum may be expected to reflect the (arbitrary) structure of the input. In this paper we shall therefore restrict our attention to the asymptotic spectrum, which may be expected to be independent of the input structure



(or, alternatively, the initial conditions). Thus we only consider the behaviour of (2.22) and (2.23) in the dissipation region.

To do this, we need only know that the spectrum falls off very rapidly in the dissipation range. Hence, from (2.22), as  $k_1 \rightarrow mk_d$  (where  $k_d$  is the Kolmogoroff dissipation wavenumber scale), we have the integral for  $v_k$  tending to zero. That is,

$$\omega_k \rightarrow vk^2 \quad \text{as } k_1 \rightarrow mk_d. \tag{2.28}$$

Similarly, in equation (2.23) for  $H(k)$ , due to the quadratic nature of the kernel, as  $k_2 \rightarrow nk_d$ , we may replace the upper limit by  $j = \infty$  and neglect the second order of small quantities. Therefore

$$H(k) \rightarrow T(k) \quad \text{as } k_2 \rightarrow nk_d. \tag{2.29}$$

Thus, the energy equation (2.26) reduces to

$$\frac{\partial q_k}{\partial t} = -2vk^2q_k + T(k), \quad \text{for } k > k_d, \tag{2.30}$$

which is just the general form (2.1), with  $h_k = 0$  in the dissipation range. Therefore it seems that our formulation goes over smoothly into the dissipation range.

In the next two sections we will show that both  $H(k)$  and  $v_k$  may be reduced to simpler forms, thus permitting us to obtain analytical solutions for the energy spectra.

### 3. A differential form of the energy equation

In the energy diffusion term  $H(k)$ , the variable  $j$  is always less than  $k$ , so we may make expansions in powers of  $j/k$ . We will retain non-vanishing terms up to, and including, second order. As an example, we write  $q_{|k+j|}$  as

$$q_{|k+j|} = q_k + \lambda q'_k + \frac{1}{2}\lambda^2 q''_k + O(\lambda^3), \tag{3.1}$$

where

$$\lambda = k \left[ \frac{j}{k}\mu + \frac{1}{2} \frac{j^2}{k^2}(1-\mu^2) + O\left(\frac{j^3}{k^3}\right) \right], \tag{3.2}$$

and a prime will be used to denote  $\partial/\partial k$ , when this is convenient.

From equation (2.23),  $H(k)$  is given by

$$H(k) = \int_{j \leq nk} d^3j \frac{2L_{kj}q_j(q_{k+\lambda} - q_k)}{\omega_k + \omega_j + \omega_{k+\lambda}}. \tag{3.3}$$

Let us call the integrand  $I$ , then using (3.1), this is written as

$$I = \frac{2L_{kj}q_j(\lambda q'_k + \frac{1}{2}\lambda^2 q''_k)}{2\omega_k(1 + \omega_j/2\omega_k + \frac{1}{2}\lambda\omega'_k/\omega_k + \frac{1}{2}\lambda^2\omega''_k/\omega_k)}. \tag{3.4}$$

As we shall see later,  $\omega_j$  varies as  $j^{2/3}$  in the inertial range and as  $j^2$  in the dissipation range. For convenience we put

$$\frac{\omega_j}{2\omega_k} = O(\lambda),$$

without serious loss of accuracy. Further, as this term makes its lowest non-vanishing contribution at  $O(\lambda^3)$ , we will suppress it altogether at this stage.

Therefore we have

$$\begin{aligned}
 I &= \frac{L_{kj}q_j}{\omega_k} \left[ \left( \lambda q'_k + \frac{\lambda^2}{2} q''_k \right) \left( 1 - \frac{\lambda}{2} \frac{\omega'_k}{\omega_k} \right) + O(\lambda^3) \right] \\
 &= \frac{L_{kj}q_j}{\omega_k} \left( \lambda q'_k + \frac{\lambda^2}{2} q''_k - \frac{\lambda^2}{2} q'_k \frac{\omega'_k}{\omega_k} + O(\lambda^3) \right) \\
 &= L_{kj}q_j \left[ \frac{\lambda^2}{2} \frac{\partial}{\partial k} \left( \frac{q'_k}{\omega_k} \right) + \frac{\lambda}{\omega_k} q_k \right] \\
 &= \frac{\partial}{\partial k} \left( L_{kj}q_j \frac{\lambda^2}{\omega_k} \frac{q_k}{2} \right) - q_j \frac{\lambda^2}{2} \left( \frac{q'_k}{\omega_k} \right) \frac{\partial}{\partial k} L_{kj} + L_{kj} \frac{\lambda}{\omega_k} q_j q'_k.
 \end{aligned} \tag{3.5}$$

From equation (2.5), we obtain for  $L_{kj}$

$$L_{kj} = k^2(1 - \mu^2) \left[ 1 - \frac{j^2}{k^2} + O\left(\frac{j^3}{k^3}\right) \right]. \tag{3.6}$$

Using this, and equation (3.2) for  $\lambda$ , we substitute (3.5) back into (3.3) and evaluate the angular integrations, thus:

$$\begin{aligned}
 H(k) &= 2\pi \int_0^{nk} j^2 dj \int_{-1}^1 d\mu \left[ \frac{\partial}{\partial k} \left( L_{kj}q_j \frac{\lambda^2}{2} \frac{q'_k}{\omega_k} \right) + L_{kj} \frac{\lambda q_j q'_k}{\omega_k} - \frac{q_j \lambda^2}{2} \left( \frac{q'_k}{\omega_k} \right) \frac{\partial L_{kj}}{\partial k} \right] \\
 &= 2 \int_0^{nk} j^2 dj \left[ \frac{\partial}{\partial k} \left( \frac{4\pi}{15} \frac{k^2}{\omega_k} j^2 q_j \frac{\partial q_k}{\partial k} \right) + \frac{2}{k} \left( \frac{4\pi}{15} \frac{k^2}{\omega_k} \frac{\partial q_k}{\partial k} j^2 q_j \right) \right].
 \end{aligned} \tag{3.7}$$

Using the relationship

$$\int_0^{nk} dj \frac{\partial}{\partial k} = \frac{\partial}{\partial k} \int_0^{nk} dj - \int_0^\infty dj n \delta(nk - j),$$

where  $\delta$  is a Dirac delta function, equation (3.7) becomes

$$\begin{aligned}
 H(k) &= \left( \frac{\partial}{\partial k} + \frac{2}{k} \right) \int_{j \leq nk} j^2 dj \frac{4\pi}{15} \frac{k^2}{\omega_k} j^2 q_j \frac{\partial q_k}{\partial k} - \int_0^\infty j^2 dj \frac{4\pi}{15} \frac{k^2}{\omega_k} j^2 q_j n \delta(nk - j) \frac{\partial q_k}{\partial k} \\
 &= \left( \frac{\partial}{\partial k} + \frac{2}{k} \right) \left( A_2(k) \frac{\partial q_k}{\partial k} \right) - A_1(k) \frac{\partial q_k}{\partial k}.
 \end{aligned} \tag{3.8}$$

The coefficients are readily seen to be

$$A_1(k) = \frac{k^2}{15\omega_k} \int_0^\infty dj n \delta(nk - j) j^2 E(j) \tag{3.9}$$

and

$$A_2(k) = \frac{k^2}{15\omega_k} \int_0^{nk} dj j^2 E(j), \tag{3.10}$$

where we have invoked the relationship  $E(k) = 4\pi k^2 q_k$ , in order to put  $A_1$  and  $A_2$  in terms of the energy spectrum.

We complete the work of this section by forming the equation for the energy spectrum. Substituting (3.8) into equation (2.26) yields for  $q_k$

$$\frac{\partial q_k}{\partial t} = h_k - 2\omega_k q_k + \left(\frac{\partial}{\partial k} + \frac{2}{k}\right) \left(A_2(k) \frac{\partial q_k}{\partial k}\right) - A_1(k) \frac{\partial q_k}{\partial k}. \tag{3.11}$$

Then, multiplying both sides of (3.11) by  $4\pi k^2$ , we obtain

$$\frac{\partial E(k)}{\partial t} = 4\pi k^2 h_k - 2\omega_k E(k) + \frac{\partial}{\partial k} \left(k^2 A_2(k) \frac{\partial E(k)}{\partial k}\right) - k^2 A_1(k) \frac{\partial E(k)}{\partial k}, \tag{3.12}$$

for the energy spectrum.

In the next section we will obtain an approximate equation for  $\omega_k$ , which will allow (3.12) to be solved analytically.

#### 4. A simplified equation for the effective viscosity

The integrand for the effective viscosity may also be expanded, but this time we consider powers of  $k/j$ , and again retain second-order terms. For example, we may expand  $q_{|k+j|}$  in the form

$$q_{|k+j|} = q_j + \gamma q'_j + \frac{1}{2} \gamma^2 q''_j + O(\gamma^3), \tag{4.1}$$

where primes now denote  $\partial/\partial j$ , when this is convenient. The expansion parameter  $\gamma$  is given by:

$$\gamma = j \left[ \frac{k\mu}{j} + \frac{1}{2} \frac{k^2}{j^2} (1 - \mu^2) + O\left(\frac{k^3}{j^3}\right) \right]. \tag{4.2}$$

Let us now write equation (2.22) for the effective viscosity, in the form

$$k^2 \nu_k = - \int_{j \geq mk} d^3 j \frac{L_{kj} q_j (q_{j+\gamma} - q_k)}{q_k (\omega_k + \omega_j + \omega_{j+\gamma})}. \tag{4.3}$$

Expanding  $q_{j+\gamma}$  first, gives

$$k^2 \nu_k = \int_{j \geq mk} d^3 j \frac{L_{kj}}{\omega_k + \omega_j + \omega_{j+\gamma}} \left[ q_j - \frac{q_j^2}{q_k} - \frac{\gamma q_j q'_j}{q_k} - \frac{\gamma^2}{2} \left( \frac{q_j q''_j}{q_k} \right) \dots \right]. \tag{4.4}$$

For  $j > k$ , we have  $q_j < q_k$  and in particular, in the inertial range,  $q_j/q_k \sim O(\gamma^{11/3})$ . Thus we need retain only the first term on the left-hand side of (4.4): we put

$$k^2 \nu_k = \int_{j \geq mk} d^3 j \frac{L_{kj} q_j}{\omega_k + \omega_j + \omega_{j+\gamma}}. \tag{4.5}$$

and this may be compared to equation (2.7), for  $R(k)$ .

Expanding  $\omega_{j+\gamma}$  in powers of  $\gamma$ , equation (4.5) becomes

$$\begin{aligned} k^2 \nu_k &= \int_{j \geq mk} d^3 j \frac{L_{kj} q_j}{\omega_k + 2\omega_j + \gamma \omega'_j + \frac{1}{2} \gamma^2 \omega''_j + \dots} \\ &= \int_{j \geq mk} d^3 j \frac{L_{kj} q_j}{2\omega_j \left( 1 + \frac{1}{2} \omega_k/\omega_j + \frac{1}{2} \gamma \omega'_j/\omega_j + \dots \right)}. \end{aligned} \tag{4.6}$$

As in the previous section, we treat  $\omega_k/2\omega_j$  as  $O(\gamma)$ : again, we find this term makes its

lowest non-vanishing contribution in the third order. Accordingly, for manipulative convenience we will drop it at this stage. Also, from (2.5) we may show that

$$L_{kj} = -j^2(1 - \mu^2) \left[ \frac{k\mu}{j} - \frac{2k^2\mu^2}{j^2} + O\left(\frac{k^3}{j^3}\right) \right] \tag{4.7}$$

and as this has lowest order  $O(\gamma)$ , we need only retain terms which are  $O(\gamma)$  in the denominator of (4.6).

Equation (4.6) is then written as

$$k^2 v_k = \int_{j \geq mk} d^3j \frac{L_{kj} q_j}{2\omega_j} \left( 1 - \frac{\gamma}{2} \frac{\omega_j'}{\omega_j} \right) = \pi \int_{mk}^{\infty} j^2 dj \int_{-1}^1 d\mu q_j L_{kj} \left( 1 - \frac{\gamma}{2} \frac{\omega_j'}{\omega_j} \right), \tag{4.8}$$

and, with the explicit substitution of  $L_{kj}$  and  $\gamma$ , the angular integration may be performed to yield

$$k^2 v_k = \frac{8\pi}{15} k^2 \int_{mk}^{\infty} dj \frac{j^2 q_j}{\omega_j} \left( 1 + \frac{1}{4} \frac{j}{\omega_j} \frac{\partial \omega_j}{\partial j} \right). \tag{4.9}$$

This may be put in terms of the energy spectrum,

$$k^2 v_k = \frac{2}{15} k^2 \int_{mk}^{\infty} dj \frac{E(j)}{\omega_j} \left( 1 + \frac{1}{4} \frac{j}{\omega_j} \frac{\partial \omega_j}{\partial j} \right) \tag{4.10}$$

and, finally, we write

$$\omega_k = (v + v_k)k^2 = vk^2 + \frac{2}{15} k^2 \int_{mk}^{\infty} dj \frac{E(j)}{\omega_j} \left( 1 + \frac{1}{4} \frac{j}{\omega_j} \frac{\partial \omega_j}{\partial j} \right). \tag{4.11}$$

Thus equations (3.12) and (4.11) are the necessary two equations for the energy spectrum  $E(k)$  and the total viscosity (or modal lifetime)  $\omega_k$ .

### 5. Calculation of the energy spectrum

#### 5.1. The inertial region

Let us introduce a new quantity, the flux of energy which we will call  $\epsilon(k)$ . If we write this as

$$\epsilon(k) = -k^2 A_2(k) \frac{\partial E(k)}{\partial k} \tag{5.1}$$

then equation (3.12) may be written as

$$\frac{\partial E(k)}{\partial t} + \frac{\partial \epsilon(k)}{\partial k} = -2\omega_k E(k) - k^2 A_1(k) \frac{\partial E(k)}{\partial k} \tag{5.2}$$

This equation may be interpreted as:

- rate of change of energy flux = the diffusive input to mode  $k$
- the dissipation due to viscosity and small eddies.

In the inertial range we have, by definition, a balance between the latter two, so that

$$\frac{\partial E(k)}{\partial t} + \text{div } \epsilon(k) = 0, \quad (5.3)$$

where ( $\text{div} \equiv \partial/\partial k$ ) and for the steady state

$$\text{div } \epsilon(k) = 0. \quad (5.4)$$

The energy balance is then given by

$$k^2 A_1(k) \frac{\partial}{\partial k} \frac{E(k)}{k^2} + 2\omega_k E(k) = 0. \quad (5.5)$$

From (5.4), we have the result appropriate to the inertial range

$$\epsilon(k) = \text{constant} = \epsilon, \quad (5.6)$$

where  $\epsilon$  is the dissipation rate (see § 1). Hence, from (5.1) the inertial range spectrum satisfies

$$-k^2 A_2(k) \frac{\partial}{\partial k} \frac{E(k)}{k^2} = \epsilon \quad (5.7)$$

and from equation (4.11), the inertial range form of  $\omega_k$  satisfies

$$\omega_k = \frac{2}{15} k^2 \int_{mk}^{\infty} dj \frac{E(j)}{\omega_j} \left( 1 + \frac{1}{4} \frac{j}{\omega_j} \frac{\partial \omega_j}{\partial j} \right). \quad (5.8)$$

Direct substitution shows that (5.7) and (5.8) are satisfied by (see § 1)

$$\begin{aligned} E(k) &= \alpha \epsilon^{2/3} k^{-5/3} \\ \omega_k &= \beta \epsilon^{1/3} k^{2/3}, \end{aligned} \quad (5.9)$$

where  $\alpha$  and  $\beta$  are constants. The inertial range form of (5.8) becomes

$$\beta = \left(\frac{7}{60}\right)^{1/2} \alpha^{1/2} m^{-2/3} \quad (5.10)$$

and substitution of this result and (5.9) into equations (3.9) and (3.10) further yields

$$A_1(k) = \left(\frac{4}{105}\right)^{1/2} \alpha^{1/2} \epsilon^{1/3} m^{-2/3} k^{5/3} \quad (5.11)$$

$$A_2(k) = \left(\frac{3}{140}\right)^{1/2} \alpha^{1/2} \epsilon^{1/3} m^{-2/3} k^{8/3}. \quad (5.12)$$

Making use of (5.9) and (5.12), equation (5.7) becomes

$$\epsilon = \left(\frac{3}{140}\right)^{1/2} \left(\frac{11}{3}\right) \alpha^{3/2} \epsilon m^{-2/3} \quad (5.13)$$

and hence, for the Kolmogoroff constant,

$$\alpha = 1.5 m^{4/9}. \quad (5.14)$$

The terms in the energy balance (5.5) may also be evaluated. From (5.9), (5.10) and (5.14)

$$2\omega_k E(k) = -2\alpha \beta \frac{\epsilon}{k} = -\frac{14}{11} \frac{\epsilon}{k}, \quad (5.15)$$

while

$$k^2 A_1(k) \frac{\partial}{\partial k} \frac{E(k)}{k^2} = \frac{4}{3} \frac{\epsilon}{k}. \quad (5.16)$$

Substituting these results into (5.5) yields

$$k^2 A_1(k) \frac{\partial}{\partial k} \frac{E(k)}{k^2} + 2\omega_k E(k) - \Delta(k) = 0, \quad (5.17)$$

where  $\Delta(k)$ , the difference between (5.15) and (5.16), is given by

$$\Delta(k) = \frac{2}{33} \frac{\epsilon}{k}. \quad (5.18)$$

Thus the energy terms balance to within 5%, which seems quite an acceptable level of accuracy.

### 5.2. The dissipation region

At very large wavenumbers, we may expect the kinematic viscosity to play an increasing part in the energy balance. Under these circumstances, the energy flux concept is no longer helpful and we must use the energy equation, as given by (3.12). The wavenumber at which viscous effects become important is normally taken to be given (in order of magnitude) by the Kolmogoroff dissipation wavenumber,

$$k_d = \left( \frac{\epsilon}{\nu^3} \right)^{1/4} \quad (5.19)$$

which is the only form that can be obtained by dimensions from  $\epsilon$  and  $\nu$ .

It is known from experiment (Gibson and Schwarz 1963) that  $E(k)$  falls off faster than a power law in the dissipation region. We will shortly show that

$$E(k) \sim \exp(-k^2/k_d^2)$$

but first we consider the effect of this on the various coefficients in the theory.

Physically, we expect to have  $\nu_k < \nu$  in the dissipation region. Consider equation (4.10) for the effective viscosity: as  $k$  becomes greater than  $k_d$ ,  $E(k)$  rapidly becomes small, and tends to zero. Hence we have,

$$\nu_k \rightarrow 0 \quad \text{as } k \rightarrow \infty,$$

and as a consequence we can write

$$\omega_k \rightarrow \nu k^2 \quad \text{for } k > k_d. \quad (5.20)$$

From (3.9) we have

$$A_1(k) = \frac{k^2}{15\omega_k} n^3 k^2 E(nk)$$

and substituting from (5.20), this becomes

$$A_1(k) = \frac{1}{15\nu} n^3 k^2 E(nk). \quad (5.21)$$

The second coefficient is given by (3.10), thus

$$A_2(k) = \frac{k^2}{15\omega_k} \int_0^{nk} dj j^2 E(j) \quad (5.22)$$

and for  $E(j)$  an exponentially decreasing function at large  $j$ , we may write the integral in (5.22) as

$$\int_0^{nk} dj j^2 E(j) \simeq \int_0^\infty dj j^2 E(j) = \frac{\epsilon}{2\nu}, \quad (5.23)$$

the last step following from the definition of the dissipation rate,  $\epsilon$ . Then substituting (5.23) and (5.20) into (5.22),  $A_2(k)$  becomes

$$A_2(k) \equiv A_2 = \frac{\epsilon}{30\nu^2}. \quad (5.24)$$

Hence, with the substitution of (5.20), (5.21) and (5.23) into (3.12), the steady-state energy equation takes the dissipation range form:

$$\frac{\epsilon}{30\nu^2} \frac{\partial}{\partial k} \left( k^2 \frac{\partial E(k)}{\partial k} \frac{1}{k^2} \right) - \frac{n^3 k^2}{15\nu} E(nk) \frac{\partial E(k)}{\partial k} \frac{1}{k^2} - 2\nu k^2 E(k) = 0. \quad (5.25)$$

The second term is explicitly quadratic in  $E(k)$ , so we may drop this, with exponentially small error. Substituting from (5.19) for  $k_d$ , equation (5.25) then takes the form

$$\frac{k_d^4}{60} \frac{\partial}{\partial k} \left( k^2 \frac{\partial E(k)}{\partial k} \frac{1}{k^2} \right) - k^2 E(k) = 0 \quad (5.26)$$

which has the asymptotic solution

$$E(k) \sim \exp \left( -\frac{\sqrt{60}}{2} \frac{k^2}{k_d^2} \right). \quad (5.27)$$

This is consistent with our dissipation range approximations. Exponential decays of the general form  $\exp(-k^2/k_d^2)$  have previously been found by Townsend (1951), Edwards and McComb (1971) and Nakano (1972). We discuss the latter theory in the next section.

## 6. Comparison with Nakano's theory and with experiment

In a recent paper (Nakano 1972) a theory has been put forward, which, although following a different analytical path from the present theory, is based on similar principles and yields very similar results.

For a given  $u_x(\mathbf{k}, t)$ , Nakano divides the nonlinear term into contributions from wavenumbers larger than  $\mathbf{k}$  and wavenumbers smaller than  $\mathbf{k}$ , thus:

$$\left( \frac{\partial}{\partial t} + \nu k^2 \right) u_x(\mathbf{k}, t) = \left( \sum_{j \geq m\mathbf{k}} + \sum_{j \leq n\mathbf{k}} \right) M_{\alpha\beta\gamma}(\mathbf{k}) u_\beta(\mathbf{j}, t) u_\gamma(\mathbf{k} + \mathbf{j}, t). \quad (6.1)$$

He argues that the effect of small eddies ( $j \geq m\mathbf{k}$ ) is a net energy loss from mode  $\mathbf{k}$  and may be characterized by an effective viscosity. The effect of large eddies ( $j \leq n\mathbf{k}$ ) is then supposed to be a net energy gain by mode  $\mathbf{k}$ , this being due to a diffusive flow of energy to large wavenumbers.

Nakano begins by integrating (6.1) forward in time, with  $j \geq m\mathbf{k}$ . An iterative method of solution (equivalent to Kraichnan's) leads to an equation for the response function  $g_{\mathbf{k}}(t)$ , which is identical to that derived by Kraichnan (1959). Further assumptions, namely: that

$$g_{\mathbf{k}}(t) = \exp(-\omega_{\mathbf{k}} t) = \exp[-(\nu + \nu_{\mathbf{k}}) k^2 t] \quad (6.2)$$

and that  $q_k(t)$  damps slowly compared to  $g_k(t)$ , followed by an expansion of the wave-number integrand in powers of  $k/j$ , lead to

$$\left(\frac{\partial u_\alpha(\mathbf{k}, t)}{\partial t}\right)_{\text{small eddies}} = \sum_{j \geq mk} M_{\alpha\beta\gamma}(\mathbf{k}) u_\beta(\mathbf{j}, t) u_\gamma(\mathbf{k} + \mathbf{j}, t) = -v_k k^2 u_\alpha(\mathbf{k}, t), \tag{6.3}$$

where

$$v_k = \frac{16\pi}{15} \int_{mk}^\infty dj \frac{E(j)}{j^2 v_j} \left( 1 + \frac{1}{j^2 v_j} \frac{j}{2} \frac{\partial}{\partial j} (j^2 v_j) \right). \tag{6.4}$$

The effect of large eddies is then to be found from

$$\left(\frac{\partial u_\alpha(\mathbf{k}, t)}{\partial t}\right)_{\text{large eddies}} = \sum_{j \leq nk} M_{\alpha\beta\gamma}(\mathbf{k}) u_\beta(\mathbf{j}, t) u_\gamma(\mathbf{k} + \mathbf{j}, t). \tag{6.5}$$

The velocity  $u_\gamma(\mathbf{k} + \mathbf{j}, t)$  is expanded as

$$u_\gamma(\mathbf{k} + \mathbf{j}, t) = u_\gamma(\mathbf{k}, t) + \mathbf{j} \cdot \nabla_k u_\gamma(\mathbf{k}, t) + \dots \tag{6.6}$$

and substituted into (6.5).

Nakano argues that the first term in (6.6) gives rise to a rather spurious dissipative term, which is relatively inefficient compared to the small eddy effects discussed earlier. He neglects this term but it is worthy of note that the analogous term in the present theory (see equations (3.1), (3.3) and (3.4)) vanishes by explicit cancellation. He retains only the second term in (6.6), which he calls the gradient coupling term.

Again, an iterative method of solution is used. Equation (6.5), with only the gradient term retained, leads to

$$\left(\frac{\partial u_\alpha(\mathbf{k}, t)}{\partial t}\right)_{\text{large eddies}} = D_{\alpha\beta}(k) \left( \frac{\partial}{\partial k_\gamma} B_{\gamma\sigma}(\mathbf{k}) \frac{\partial}{\partial k} - C_\sigma(k) \frac{\partial}{\partial k_\sigma} \right) u_\beta(\mathbf{k}, t), \tag{6.7}$$

where the coefficients are given by

$$k_\gamma k_\sigma B_{\gamma\sigma}(k) = B(k) = \frac{2}{15v_k} \int_0^{nk} dj j^2 E(j) \tag{6.8}$$

and

$$k_\sigma C_\sigma(k) = C(k) = \frac{2}{15v_k} \int_0^\infty dj n \delta(nk - j) j^2 E(j) \tag{6.9}$$

(note: we use  $B_{\gamma\sigma}$  rather than Nakano's  $D_{\gamma\sigma}$ , to avoid confusion with our own notation for the projection operator).

The cut-off ratios are then related by the principle of action and reaction: it is to be expected that, just as an eddy with wavenumber  $\mathbf{j}$  acts on mode  $\mathbf{k}$  as an effective viscosity, an eddy with wavenumber  $\mathbf{k}$  distorts an eddy  $\mathbf{j}$  with an equal reaction. It follows that the conditions  $\mathbf{j} \geq m\mathbf{k}$  and  $n\mathbf{j} \geq \mathbf{k}$  should be equivalent, or

$$m = n^{-1}. \tag{6.10}$$



Finally, the energy equation is formed using (6.1), (6.3) and (6.7) to yield

$$\frac{\partial}{\partial t} E(k) = \frac{\partial}{\partial k} \left( B(k) k^2 \frac{\partial}{\partial k} \frac{E(k)}{k^2} \right) - 2\omega_k E(k) - C(k) k^2 \frac{\partial}{\partial k} \frac{E(k)}{k^2} + \left( \frac{L}{2\pi} \right)^3 4\pi k^2 B_{\gamma\sigma}(k) \left\langle \frac{\partial u_\alpha(\mathbf{k}, t)}{\partial k_\gamma} \frac{\partial u_\alpha(-\mathbf{k}, t)}{\partial k_\sigma} \right\rangle, \tag{6.11}$$

where  $v_k$  is given by (6.4), and  $B(k)$  and  $C(k)$  by equations (6.8) and (6.9).

These equations may be compared to the present theory. Beginning with (6.11) and (3.12), for the energy spectrum, we see that the two are identical in form, with the exception of the rather awkward term involving the gradients of the fluctuating velocity field, which occurs in Nakano's equation but not in ours.

Comparing equations (6.8), (6.9) for  $B(k)$ ,  $C(k)$  with equations (3.9), (3.10) for  $A_1(k)$ ,  $A_2(k)$  of the present theory, we see that the latter are a factor of two smaller, but otherwise identical. Similarly,  $v_k$  as given by equation (4.9) is a factor of two smaller than  $v_k$  given by equation (6.4).

To see how this difference arises, let us go back to equation (4.3) for  $\omega_k$ . The denominator is expanded in powers of  $\gamma$ , thus :

$$\frac{1}{\omega_k + \omega_j + \omega_{j+\gamma}} = \frac{1}{\omega_k + \omega_j + \omega_j + \gamma\omega'_j + \dots} = \frac{1}{2} \frac{1}{\omega_j} \left( 1 - \frac{1}{2} \frac{\omega_k}{\omega_j} - \frac{1}{2} \frac{\gamma\omega'_j}{\omega_j} + \dots \right). \tag{6.12}$$

The form of this denominator arises from an underlying assumption that

$$g_k(t) \sim e^{-\omega_k t}, \quad q_k(t) \sim e^{-\omega_k t}.$$

If we assume this is not the case but (say)

$$q_k(t) \sim e^{-\eta_k(t)}$$

then this denominator would be  $(\eta_k + \eta_j + \omega_{j+\gamma})$ .

Now Nakano assumed, in deriving his equation for the effective viscosity, that

$$\eta_k \ll \omega_k.$$

With this assumption, the denominator becomes  $\omega_{j+\gamma}$  and hence its expansion is just

$$\frac{1}{\omega_{j+\gamma}} = \frac{1}{\omega_j + \gamma\omega'_j} = \frac{1}{\omega_j} \left( 1 - \frac{\gamma\omega'_j}{\omega_j} + \dots \right). \tag{6.13}$$

Thus the factors of  $\frac{1}{2}$  arise from the use of (6.12) rather than (6.13).

Not surprisingly, both theories yield the Kolmogoroff distribution in the inertial range and the same form of exponential decay in the dissipation range.

The quantitative aspect may be checked by calculating the Kolmogoroff constant. From Nakano's theory one has

$$\alpha = 1.3m^{4/9}, \tag{6.14}$$

whereas the present theory gives

$$\alpha = 1.5m^{4/9}. \tag{5.14}$$

Nakano arbitrarily takes values of  $m = 2, 3$  and  $4$  to compare his theory with experiment. Doing this, we obtain the following comparative table.

**Table 1.** Theoretical values of the Kolmogoroff constant.

	$m = 2$	$m = 3$	$m = 4$
Nakano (1972)	1.7	2.1	2.3
Present theory	2.0	2.4	2.6

These results may be compared with some experimental results in the following table.

**Table 2.** Experimental values of the Kolmogoroff constant.

Kistler and Vrebalovitch (1961)	Grant <i>et al</i> (1962)	Gibson (1963)	Gibson and Schwartz (1963)
2.7	1.4	1.6	1.3

Thus although the theoretical values of  $\alpha$  are rather on the large side (that is, of course, assuming that  $m$  actually is as large as 2: we shall discuss this point later) they fall sufficiently well within the experimental range of values to be taken seriously.

## 7. Conclusions

In this work we have taken, as our starting point, an energy equation of the Edwards (or Kraichnan) type. We have interpreted it in terms of a Heisenberg-type effective viscosity and the result has been a new equation for the latter quantity. It should, perhaps, be emphasized that the arguments which lead to equation (2.22) require only that the Reynolds number should be large enough for an inertial range to exist: apart from that, they are both rigorous and general. The only arbitrary step is the interpretation of equation (2.66) in terms of  $v_k$ : that is, equation (2.20). However, this interpretation, once made, looks so physically obvious and correct that we have few qualms about it.

In §§ 5 and 6, we believe we have demonstrated the adequacy of our equation for the effective viscosity: we find that it leads to equations which, in particular, yield the Kolmogoroff distribution and, in general, are in quite good agreement with experiment.

One question that has not been settled in this paper is the value to be taken for  $m$ . Just before equation (2.24) we noted that  $m$  was determined by the rate of cancellation of the integrand in (2.22), for  $k \sim j$ . Thus one could quite reasonably obtain  $m$  from the value of  $k/j$  when the integrand drops to (say) 5% of its peak value.

As the terms involved are quite rapidly varying functions of wavenumber it seems unlikely that  $m$  is any greater than two. As the lower bound on  $m$  is unity we might suppose that we are faced with  $1 \leq m \leq 2$  and, from equation (5.14),  $1.5 \leq \alpha \leq 2.0$ , for the Kolmogoroff constant. In fact, a very crude numerical estimate suggests  $m \approx 1.03 = 1.0$  (to our level of accuracy). While this must not be taken too seriously, it does raise the question of the convergence of the expansions in §§ 3 and 4.

Taking the expansion, in powers of  $j/k$ , which occurs in § 3, we have, as the worst case  $j/k = 1/m$ . If  $m \sim 1$ , then the expansion is not convergent for all  $j/k$ . However, one must remember that the expansion is performed inside the integral, and hence one may well have convergence 'in the mean'.

Another factor involved, is that both expansions (§ 3 and § 4) are coupled by the joint presence of  $\omega$  and  $q$ : this also makes it difficult to pronounce in any definite or obvious way on their performance under integration, when  $m \sim 1$ . In this paper we have taken the view that 'point by point' convergence will be adequate with  $m \sim 2$ , and that convergence 'in the mean' may be very much better. We think the results obtained support the latter supposition.

In any case, it is hoped to settle questions of this kind, and evaluate  $m$ , as part of a more general numerical analysis of the theory presented. This will be the subject of further work.

Our final point concerns the energy equation, which was our starting point. This equation has been derived by various methods, but, irrespective of the method used, certain classes of terms in an expansion are summed to all orders while other terms are neglected. To date it has not been shown that higher order terms in the closure expansion for the energy are small in comparison to the terms retained in (say) equation (2.1). It seems reasonable to suppose that the relatively simple analytical techniques of the present paper may offer a useful method of attacking this problem.

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